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A short theory of the error process

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ABSTRACT

In spite of endless discussions about uncertainties the debate about errors is less animated. It makes sense to define a general error process as a sub-process within any erroneous real world process. This can be done in a top down concept by an appropriate model, usable in any practical application with the help of scientific adaptation. The following introduction is primarily based on theoretical considerations.

1. Introduction

A measurement process is not the only erroneous and uncertain process. Unfortunately, errors and uncertainties prosper, overarch and sustain anywhere. Well, for such common phenomena a common investigation strategy in form of an individual *error process* and of an individual *uncertainty process* should be self-evident: Especially so in Metrology, which fortunately is based on sound theoretical tools like Signal and System Theory, Stochastics and Statistics as well as particular branches of Mathematics, like Linear Algebra and Differential Calculus. Nevertheless, in spite of such an abstract appearance, the concept of error and uncertain structures is a relatively simple topic, if systematically approached.

The following sections provide a holistic proposal for the model of a dedicated, self-contained *sub-process*, within an erroneous process, consequently called *error process*. In order to avoid the usual mix of error and uncertainty issues, only error concerns are taken into detailed account here. It has to be accepted that the terms *error* and *uncertainty* have quite a different meaning: Errors are properties of processes, and processes are not uncertain. Uncertainties are properties of our *knowledge*. On the other hand, it has to be admitted too that the strategies of uncertainty modelling and analysis are quite similar to the strategies of error modelling and analysis. But, uncertainty modelling and analysis presuppose and follow error modelling and analysis as indispensable *apriority knowledge*. Otherwise, uncertainty statements are not competent to incorporate information about errors, *knowable* or *unknowable*. If for example random quantities are effective *in* or *on* a process, the resulting *random error quantities* [2] have to be explored numerically by their *actual characteristic values* first. Only if there are doubts about these values, *uncertainty quantities* with their values concerning our momentary knowledge (*degree of belief*, *confidence*) come into focus.

The following procedure is configured to such an extent that the basic concept of an error process is generally valid and thus largely independent from applicational demands. It is clear that in modelling practice on the one hand many types of quantity models turn up: *constant* and *dependent*, *continuous* and *discrete*, *time* and *frequency conditioned*, *deterministic* and *random*, *monovariate* and *multivariate*. On the other hand, processes are diverse as well. They are modelled as *linear* or *nonlinear*, *nondynamic* or *dynamic*, *time* and *frequency dependent*, and so on. Nevertheless, any top down concept must be independent of such particular features, and the chosen concept has to be subsequently adaptable, more or less straightforwardly, to individual requests in practice. Thus, the following proposal assumes continuous, time dependent, deterministic, multivariate quantities together with linear, time independent (LTI), stable, multivariate dynamic processes.

For the description of dynamic processes, including the dynamic error and uncertainty sub-processes, the well-known State Description (SD) method of Signal and System Theory has been chosen [3,4]. The main advantage is the provision of an omnipotent structure of a process model. Besides, all inner *state quantities* of a process model are incorporated in this *input-state-output* structure, in spite of the conventional, fragmentary *input-output* configuration. Moreover, a dynamic process model can easily be reduced to a nondynamic process model in *steady state* condition, just by trivial intervention.

Now to terminology: The situation in Metrology is not satisfactory; one still has to choose one's individual concept. In the following sections, *real world items* are called *processes*, and their active or passive *instances* are called *procedures*. On the other hand, processes are characterised by *real world items*, here called *quantities* and their real *interrelations*. Now, in addition to this *real world domain*, we enter the related *abstract world domain of process models* and *quantity models*. This *model domain* is abstract due to our abstract ideas and imagination: The model of a process is not the process, and the model of a quantity is not the quantity. We have a duality of a real world and an abstract world, the abstract world being the model of the real world.

Signal and System Theory (SST), in fact a *mathematical* tool [6], and thus effective in the model domain, covers abstract items. Accordingly, we call the models of the objective real world items *signals* and *systems*. These terms are well-known, but arbitrarily used anywhere. The *annotation* of quantities and processes is adopted for signals and systems.

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Signal and System Theory:

An abstract signal is the model of a real world quantity.

An abstract system is the model of a real world process.

Thus, quantities and processes can be represented by mathematical expressions and visualized by equivalent Signal Relation Graphs (SRG) [5]. Remarkably, in the abstract model domain, systems (models of processes) show up “only” as interrelations between selected and defined signals.

Here, processes are meant to be general, real world processes. Thus a measurement process is just one of them. Errors, uncertainties and parameters are considered quantities and thus need equivalent signals as their models. Accept that Signal and System Theory pursue equal concepts for all types of signals and systems, a very welcome situation.

2. Remarks concerning error processes

The assumption of a common *error process* model should be helpful to clarify the current stagnant and rather vague error and uncertainty discussion [8,9]. First of all, the model of an error process should be located within the model of an erroneous, real world process, as soon as the existence of *error quantities*, their causes and interactions are suspected (Fig. 1). Of course, should a process be error-free, which is never the case, no error process would bother us.

The model of the *uncertainty process* U is not discussed furthermore. The main question prevails, what the *structure* and the *parameters* of the *error process* model E look like (section 5) and where and how the detected *error quantities* **e**(t) impact defined quantities of interest (section 7).

The main model structure of an error process bases on the prominent *error definition*, which always presumes a check of an actual *quantity of interest* against some *nominal* (reference) *quantity* by assignment (Fig. 2):

$$e_y(t) = y_e(t) - y_{nom}(t) \quad [\{y\}]$$

or to present the additive error

$$y_e(t) = y_{ref}(t) + e_y(t) \quad [\{y\}]$$

However, if no nominal (reference) quantity is available, for whatever reason, an error quantity can’t be defined at all. This statement is generally used as an argument to even negate the very existence of error quantities, although they are mentioned in the GUM [1] as well as in the VIM [2]. If a nominal (reference) quantity exists, but only in a vague form, a precise error description cannot be developed. It may be estimated at best. This is the situation, which finally leads to an *error uncertainty quantity*. The same is true, if on the other side the actual quantity cannot be determined exactly, for example by measurement or other means. An error still exists.

This justifies the prevailing *duality* of *error quantities* and *uncertainty quantities* und does not foster the exclusivity of uncertainty quantities at all.

It is essential to first define error quantities in an idealised, hypothetically *certain-world setting*, in order to develop relevant *error model*

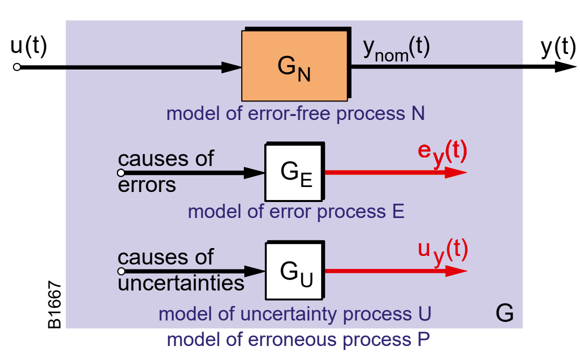

Fig. 1. Basic model of the erroneous process P, containing the sub-models of nominal, error free process N, error process E and uncertainty process U. (G is the general transfer function symbol).

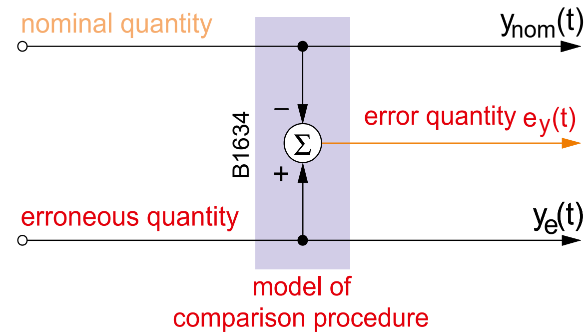

Fig. 2. Definition of the error quantity (error quantity $e_y(t)$; nominal quantity $y_{nom}(t)$; erroneous quantity $y_e(t)$).

structures and *error parameter quantities* within the model of the *error process* E (Figs. 1 and 6).

But the analytical and empirical transition to uncertain circumstances with the emergence of *uncertain error quantities*, and therefore of *uncertainty quantities*, will naturally lead to a more demanding *uncertainty process* U (not treated here), in which the particular *numerical values* of the defined error quantities will be uncertain to some degree (Fig. 1).

Influences of error quantities may be corrected [1] and the unavoidable uncertainty quantity values have to be stated. As a vision, we are also allowed to presume that errors from random influence quantities may be corrected in a certain-world setting and even, at least approximately, in an uncertain-world setting. This requires acknowledging the independent existence of *random error quantities* and *uncertainty quantities*.

To give a preliminary résumé: To be realistic, error quantities, induced by error processes, are always existent and active, but their individual values may be uncertain. Thus, we have to deal with *error quantities* and *error uncertainty quantities* at the same time. They may be multivariate and time dependent. The model of an error process characterises the arising error quantities, and the model of an uncertainty process characterises the arising uncertainty quantities.

In addition: Within the entire (global) model of an erroneous process there appear other uncertainty quantities concerning those quantity values, which arise outside the error process, and which are uncertain due to our insufficient knowledge of them (degree of belief, confidence). All these different uncertainty types are combined in order to get a final uncertainty vector for the erroneous process.

Again, this seemingly confusing situation requires a consequent definition and distinction of error quantities and uncertainty quantities. The following sections consider the error process only.

3. Basic model of an error-free process

The multivariate, linear, time independent (LTI) differential state equation of $(N)^{th}$ order gives an *input-state-output* model of an error-free dynamic process P (Fig. 3):

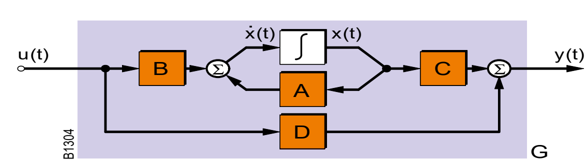

Fig. 3. Basic model of an error-free dynamic process P.

$\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ input and state signal equation

$\mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$ output signal equation

with the signal vectors

u	input	(M) elements
x	state	(N) elements
y	output	(P) elements

and with the system matrices

A	state	(N)x(N)
B	input	(N)x(M)
C	output	(P)x(N)
D	feedthrough	(P)x(M)

The mathematical *solution* of this differential equation (convolution integral equation) describes the error-free, transient behaviour (trajectory, transition, orbit, evolution) of the set of (P) output signals **y**(t), which depend continuously on the independent variable time t from $t = t_0$ onwards, on the stimulating set of (M) input signals **u**(t) and on the set of (N) initial state signals **x** (0) at time $t = t_0$:

$$\mathbf{y}(t) = \mathbf{C}\, e^{\mathbf{A}(t)}\, \mathbf{x}(0) + \mathbf{D}\, \mathbf{u}(t) + \mathbf{C} \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\, \mathbf{B}\, \mathbf{u}(\tau)\, d\tau$$

4. Basic model of an erroneous process

The minimal structure of the model of an *erroneous process* contains *input quantities* **u**(t), *disturbance quantities* **v**(t), internal *state quantities* **x** (t), *output quantities* **y**(t), and *loading quantities* **z**(t) (Fig. 4) with the extended set of model equations.

It is obvious that the *disturbance quantities* **v**(t) are *external* causes *from* the surroundings of process P, liable for the first type of errors, i.e. for *disturbance error quantities*. On the other hand, we have *loading quantities* **z**(t) acting *onto* the surroundings of process P. These may influence preceding processes, so that unwanted alterations of the input quantities **u**(t) may occur, which are liable to the second type of errors, i. e. for *loading error quantities*. The third type of errors, *internal error quantities*, becomes visible as soon as the hitherto unknown structure of the internal *error process* E shows up. Internal errors are for example transfer errors, parameter errors, model errors, time dependent errors, and so on.

At present, we assume that the model of error process E is a *linear*,

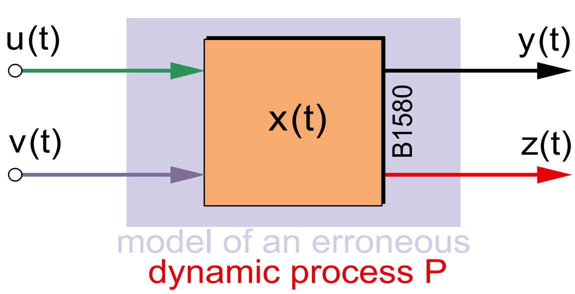

Fig. 4. Basic model of an erroneous dynamic process P, with two input quantity vectors and two output quantity vectors.

$\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) + \mathbf{E}\,\mathbf{v}(t)$
$\mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,\mathbf{u}(t) + \mathbf{F}\,\mathbf{v}(t)$
$\mathbf{z}(t) = \mathbf{G}\,\mathbf{x}(t) + \mathbf{H}\,\mathbf{u}(t) + \mathbf{J}\,\mathbf{v}(t)$

time independent (LTI) multivariate dynamic sub-model, producing *time dependent error quantities* **e**(t) as linear combinations of those three possible error types.

5. Basic model of an error process

The postulated error process E within the erroneous process P will now be covered in detail. As a generalisation we get the following *structure* (Fig. 5), where the prescribed dynamic, nominal process N may be, for example, an electronic active filter with a given *transfer function* to be realised in the frequency or in the time domain respectively. All three error types, *transfer errors*, *disturbance errors* and *loading errors* are

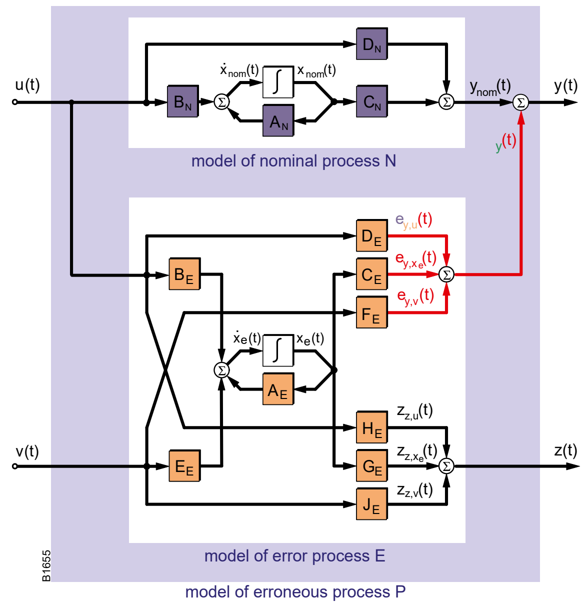

Fig. 5. Basic model of an erroneous dynamic process P, containing an error-free nominal process N and an error process E.

considered. Again, a challenge is the *identification* of the particular elements of the parameter matrices.

6. Error differential equation

By Linear Fractional Transformation (LFT) we divide the erroneous system P into two sub-systems, into the given error-free (nominal) system N and the error system E. The error system E reveals the resulting error signal vector **e**(t), here however without considering the influences of the *disturbing signal vector* **v**(t) and the *loading signal vector* **z**(t). For our endeavour we have three sets of equations:

Error-free (nominal) System N:

$$\dot{\mathbf{x}}_{nom}(t) = \mathbf{A}_N \mathbf{x}_{nom}(t) + \mathbf{B}_N \mathbf{u}(t)$$
$$\mathbf{y}_{nom}(t) = \mathbf{C}_N \mathbf{x}_{nom}(t) + \mathbf{D}_N \mathbf{u}(t)$$

Error System E:

$$\dot{\mathbf{x}}_e(t) = \mathbf{A}_E \mathbf{x}_e(t) + \mathbf{B}_E \mathbf{u}(t)$$
$$\mathbf{y}_e(t) = \mathbf{C}_E \mathbf{x}_e(t) + \mathbf{D}_E \mathbf{u}(t)$$

Error Definition:

$$\mathbf{e}_y(t) = \mathbf{y}(t) - \mathbf{y}_{nom}(t)$$

This allows us to show the *difference* between the output signal vectors $\mathbf{y}_{nom}(t)$ and $\mathbf{y}_e(t)$:

$$\begin{aligned} \mathbf{y}_e(t) &= \mathbf{C}_E\,\mathbf{x}_e(t) + \mathbf{D}_E\,\mathbf{u}(t) \\ \mathbf{y}_{nom}(t) &= \mathbf{C}_N\,\mathbf{x}_{nom}(t) + \mathbf{D}_N\,\mathbf{u}(t) \end{aligned}$$

$$\mathbf{e}_y(t) = \mathbf{y}(t) - \mathbf{y}_{nom}(t) = \mathbf{C}_E\mathbf{x}_e(t) - \mathbf{C}_N\,\mathbf{x}_{nom}(t) + (\mathbf{D}_E - \mathbf{D}_N)\,\mathbf{u}(t)$$

or as the parallel interconnection
of the two sub-systems N and E:

$$\begin{bmatrix} \dot{\mathbf{x}}_e(t) \\ \dot{\mathbf{x}}_{nom}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_E & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_N \end{bmatrix} \begin{bmatrix} \mathbf{x}_e(t) \\ \mathbf{x}_{nom}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B}_E \\ \mathbf{B}_N \end{bmatrix} \mathbf{u}(t)$$

$$\mathbf{e}_y(t) = \begin{bmatrix} \mathbf{C}_E & -\mathbf{C}_N \end{bmatrix} \begin{bmatrix} \mathbf{x}_e(t) \\ \mathbf{x}_{nom}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{D}_E \\ -\mathbf{D}_N \end{bmatrix} \mathbf{u}(t)$$

We have already got the numerical values of all elements of the process parameter matrices with errors and uncertainties by *identification* (*calibration*).

7. Error interconections

In many fields of science and technology it is assumed that error quantities superimpose in parallel connection, as has been shown so far. This is not always the case. Signal and System Theory enable three and only three connection types of two processes: The series connection, the parallel connection, and the feedback connection (Fig. 6), applicable here too.

We recognise that all three basic structures satisfy the obvious condition that the error-free nominal process N has to persist unchanged, if all error quantity influences **e**(t) of the error process E disappear.

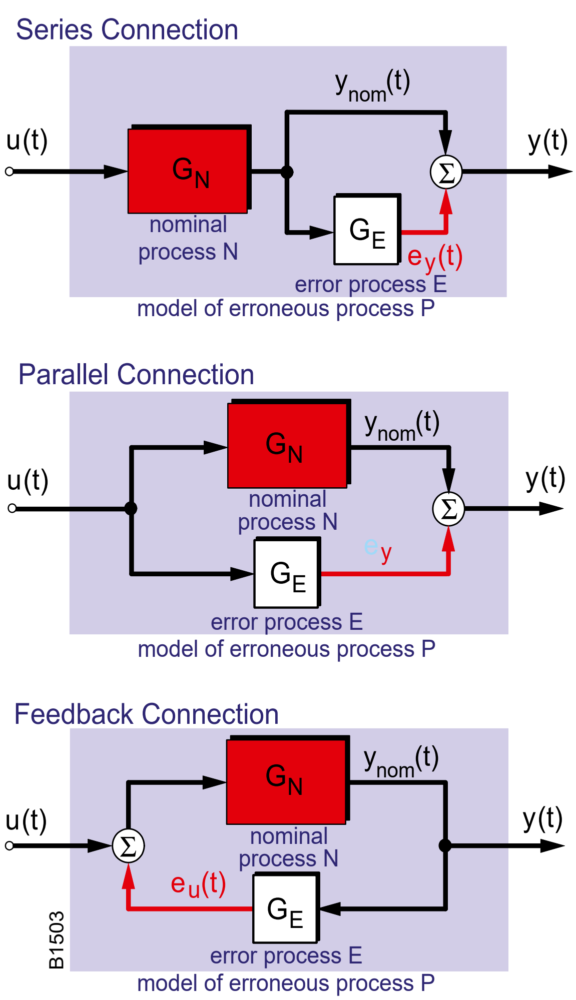

Fig. 6. Three possible interconnection models of the error process E with the nominal process N.

8. Conclusions

This is a necessarily short introduction to the important concept of error quantities as complement to the concept of uncertainty quantities, enlightening the obvious duality of the matter. It is proposed to concentrate all error relations and procedures in one error sub-process as an integral part of an erroneous process, this in parallel to an (up to now missing) uncertainty sub-process.

It is indicated that the State Description method is capable to represent the simplest error situation as well as a rather extensive and complex error structure.

The model of the error process E enables simulations together with the model of the nominal process N for further examination, especially for comparison with an empirically received data set of the global erroneous process P [7].

Further details have to be treated in the future, especially the systematic correction of error quantities (inversion, deconvolution of the error process) and the quantitative links to a sound uncertainty process, still to be defined.

Finally, it is crucial to observe that error and uncertainty concepts are not limited to applied measurement processes or to the demanding measurement tasks of National Metrology Institutes (NMI) only, they are generally valid.

References

[1] JCGM 100:2008, Guide to the Expression of Uncertainty in Measurement (GUM), 2008. Geneva.
[2] JCGM 200:2012, The International Vocabulary of Metrology (VIM), third ed., 2008. Geneva, Switzerland.
[3] S. Böhl, W. Knapp, Observer-based compensation of thermal disturbances for displacement sensors, CIRP Annals 68 (1) (2019) 543–546.
[4] S. Böhl, Error Identification and Error Correction for Spindle Test Setups Swiss Federal Institute of Technology (ETH), Zurich, PhD Thesis No. 27088, 2020.
[5] K.H. Ruhm, The signal relation diagram –A metrological tool? Measurement 127 (2018) 588–595.
[6] K.H. Ruhm, Metrology and Mathematics –Survey on a Dual Pair Advanced Mathematical and Computational Tools in Metrology and Testing XI, vol. 89, 2018, pp. 85–118.
[7] K.H. Ruhm, Measurement plus observation –A new structure in Metrology, Measurement 126 (2018) 421–432.
[8] C. Ehrlich, R. Dybkaer, Uncertainty of error:the error dilemma, NCSLI Measure 6 (3) (2011) 72–77.
[9] R. Barlow, Systematic Errors: Facts and Fictions" Advanced Statistical Techniques in HEP, The University of Manchester, 2002.

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