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Dynamics and stability – A proposal for related terms in Metrology from a mathematical point of view

Karl H. Ruhm *

Institute of Machine Tools and Manufacturing (IWF), Swiss Federal Institute of Technology (ETH), Leonhardstrasse 21, LEE-L219, CH8092 Zurich, Switzerland

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ABSTRACT

We describe measurement quantities and measurement processes verbally by using everyday terms, often sanctioned by international standards and guides. However, we apply some of these terms inconsistently as compared to definitions in base sciences like Logic and Mathematics, Signal and System Theory, Stochastics and Statistics, Estimation and Optimisation Theory. This is especially true for a group of terms in Metrology, namely for the terms *kinetic, static, time dependent, time invariant, stable, stationary, drifting* etc., which all populate the general environment of the main term *dynamics*. The paper explores systematic relations between these and similar terms and discusses the aptitude of their implementation in everyday practice. Cardinal point is the term *dynamic*, which will be investigated thoroughly. So, derived terms can be defined, incorporated and judged accordingly. For the field of Metrology some new and, maybe, surprising results arise, worth being considered in the near future.

1. Introduction

Measurement instrumentation, measurement procedures and measurement results fulfil well-defined requirements. We certify them by qualitative verbal descriptions and/or by quantitative logical and mathematical relationships, according to accepted rules. Both, qualitative and quantitative results must be equally trustworthy.

Here, the focus lies on verbal communication in Metrology, which should not just rely upon any commonly accepted habit. For example, we state colloquially that certain situations and procedures are *stable* or *unstable*. Sometimes data are reported as *drifting*. Moreover, many instances appear *nonstationary*. Are such and other descriptions identical, equivalent, synonymous, similar or related? Do they mean *dynamic*? Finally, what is the logical and/or mathematical base? Of course, these terms may serve as hints for the time being. However, what shall we do, once we have to deliver robust information? We ask for specific meaning behind these terms, regularly used in everyday metrological practice. It is obvious that such an endeavour will narrow the meaning of the everyday terms concerned and will ban some of them from the scene.

The subject area of dynamic observation, measurement and analysis is steadily increasing. This is especially true for the fields *Dynamic Identification* (Calibration), *Dynamic Measurement* and *Dynamic Reconstruction*. They all have an impact on measurement errors and measurement uncertainties. Yet, the official terminology in Metrology concerning these items stays behind on a rather low level.

We would appreciate a concept, based on systematic and coherent tools, which would be applicable to all fields of science and would thus foster top-down and holistic considerations. Indeed, there are facilities in base sciences, like Logic and Mathematics, Model Development and Identification (Calibration), Signal and System Theory,

* Tel.: +41 44 632 24 63.
E-mail address: ruhm@ethz.ch

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Stochastics and Statistics, Estimation and Optimisation [1–5,14]. However, one has to compile knowledge and skills, which are dispersed in these fundamental fields, in order to make them persuasive and thus applicable to metrological needs.

Interestingly enough the basic Guides of Metrology like the GUM [6] and the VIM [7] deliver no, or just a few helpful definitions concerning the field of consideration. Even the Standards of Statistics refrain from offering significant and precise support, when dealing with these terms [8,13].

The following sections assemble, classify and rigorously define meaningful protagonists around the term "dynamic". This is done in a structured way in order to disclose their interrelations and dependencies.

Section 2: *Signals and Systems*. We know that some of the terms in question refer to *signals* and some of them to *systems*. Therefore, we highlight the difference between the interesting real world *process* and its model, the abstract *system*, on the one hand and the difference between the interesting real world *quantity* and its model, the abstract *signal*, on the other hand.

If we are able to define and to empirically determine *properties* and *behaviour* of those abstract signals and systems, we ascribe them to the real world quantities and processes.

We consider measurement errors and measurement uncertainties to be signals too, although they have no counterpart in the real world: Nature does not know errors and uncertainties. They are defined, abstract quantities, like efficiency, comfort, sustainability and so on.

Section 3: *Description of the Dynamic System*. Naturally, the term *dynamic* is the pivot for all phenomena that involve time dependencies. What does *dynamic* actually mean? Which other terms depend on the adjective *dynamic*? When is the term *dynamic* applied wrongly? In order to clarify the situation, we will present a concise survey of this topic, based on logical and mathematical concepts: Some new aspects will appear, so that we will be able to define and classify other terms from scratch later on.

The resulting definitions are easily transferrable from the time domain to the space domain. For clarity reasons, this is not done here however.

Section 4: *Properties and Behaviour of the Dynamic System*. The important differences between the properties of a system and the behaviour of a system are highlighted. They are intuitively accepted but not properly applied in everyday practice. A signal, coming from a system, has properties too, but does it have a behaviour of its own?

Section 5: *Time Dependent Terms*. The subsequent section will concentrate on three unpopular terms in Metrology, namely *instability*, *nonstationarity* and *drift*. Here, probability concepts of Stochastics and Statistics come directly into action. Again, the question is, whether signals and/or systems are involved at least.

Summary, References and some Terms and Symbols complete the text.

Remark: Only a few remarks on common terms in everyday practice will show the variety of meaning in the diverse fields of Metrology. A complete listing would be impossible. The emphasis of this survey is placed on mathematical substantiation.

2. Signals and systems

Terms like stability, stationarity and drift are used in practice to describe properties of *signals* and of *systems* as well. However, *properties* of systems primarily show up in their *behaviour*, and their output signals are only consequences. So, one has to distinguish carefully between definitions of signals and definitions of systems.

2.1. Signals are models of quantities

A signal (state, parameter, trajectory, transition, transient, propagation, walk, motion, movement, change, evolution, solution, evolvement, development, outcome, path, course, phase, orbit, observation, history, phenomenon, data, effect, event, instant, wave, pattern, picture) *describes* verbally and/or mathematically a *natural* or *man-made* and *man-defined quantity*.

2.2. Systems are models of processes

At first sight, a system seems to describe a natural or man-made real world process; however, this is only indirectly the case. In fact, *mathematical models of processes*, which are denoted as *systems*, describe *abstract relations* and *correlations* between defined *state* and *inner quantities* of a process and defined *external quantities* according to the cause and effect principle. By external quantities, we mean *input and output signals*. Systems do not describe any physical artefact or functional interconnection: In this concept, *systems* (*models*) *are hardly process oriented*, *but rather quantity oriented* [9].

The well-known example of a *dynamic system of second order* may illustrate this seemingly uncommon perception: A mass-damper-spring process and an inductance–capacitance–resistance process share exactly the same mathematical model. Only the *Physical Units* of the particular signals and parameters, which normally do not show up in the model, move it in the neighbourhood of a dedicated physical process after all, and make it identifiable.

As a consequence, many of so-called *soft sciences*, which do not know physical artefacts, get access to the art of quantitative Model Development, because we define *systems as relations* and *correlations* between quantities only.

So, the basic concept of relations between signals concerning a dynamic system admits all types of signals, which means, models of real physical quantities, defined virtual signals without real counterparts, and errors and uncertainties as well.

Signal and System Theory contains tools that covers both, models of quantities (signals) and models of processes (systems). *State Space Description* elegantly describes a large variety of processes within the Signal and System Theory. In addition, *Linear Algebra* provides algorithms to reveal the behaviour of dynamic systems quantitatively by *Simulation*.

3. Description of the dynamic system

Concerning dynamic systems, many verbal and/or mathematical depictions are only approximately correct, although they are common usage. A qualitative statement in everyday language may serve as an example: "Dynamic measurement . . ., where (a) physical quantity being measured varies with time and where this variation may have (a) significant effect on (a) measurement result . . . and the associated uncertainty" [12]. Indeed, we need a definition on a mathematical basis, which has to be more precise.

3.1. Verbal description

Another more abstract, qualitative statement in everyday language could be: "A *dynamic system* is the description of a *process*, which evolves, changes its state, in time and space." Or: "A *dynamic system* is a *system*, whose present *state* depends on both, present and past *values* and/or present and past *derivative values* (velocity value, gradient value) of at least one *state signal*."

We ascribe the term "dynamic" to such a process. A set of related *state signals* represents the *state of a dynamic system* as part of its internal *model structure*.

Let us define the acronym of "dynamic": "A *nondynamic system* is the description of a *process*, whose *inner signals* merely depend on the present *values* of all signals concerned."

A system may consist of several nondynamic and dynamic subsystems. However, the whole system is dynamic, as soon as one subsystem is dynamic.

3.2. Mathematical description

If we describe a multivariable *process* by a *set* of coupled equations, and at least one of them is a *differential* or *difference equation* (and/or *inequality*) in time and/or space, the developed *mathematical model* is a *dynamic system*. We ascribe the term "dynamic" to such a process. Obviously, these differential (difference) equations describe the *relations* between defined *models of time independent and time dependent quantities* of a process. Again, in this framework they do not describe physical artefacts.

Differential equations of higher than second order rarely occur during model development. If they do all the same, we are able to convert them into a set of (N) *coupled homogeneous differential equations of first order* by defining the respective derivative terms of the equations as new signals $x_n(t)$, which we call *state signals* and which we compound in the *state signal vector* $\mathbf{x}(t)$ [2,4]:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \\ \vdots \\ \dot{x}_N(t) \end{bmatrix} = \begin{bmatrix} f_1\{\mathbf{x}(t);t\} \\ f_2\{\mathbf{x}(t);t\} \\ \vdots \\ f_n\{\mathbf{x}(t);t\} \\ \vdots \\ f_N\{\mathbf{x}(t);t\} \end{bmatrix} = \mathbf{f}\{\mathbf{x}(t);t\} \tag{1}$$

with the (N) *nonlinear, scalar functions* $f_n\{\ldots;t\}$ and the *vector function* $\mathbf{f}\{\ldots;t\}$. Whether the individual state signals $x_n(t)$ are of physical significance or of no real significance is irrelevant.

This set of equations marks the *core* (kernel) of a nonlinear, dynamic system of finite (N)th order and is the *intrinsic model* of the dynamic process, called *state model*. It fully represents the "*dynamics*" of the system with respect to the continuous time *t*.

In a further step, the so-called *State Space Description* shows relations to *external quantities*. This is elegantly done in the multivariable *input-state-output model*. It *relates*

- a defined *input signal vector* $\mathbf{u}(t)$ (input space; (M) elements)
- the given *state signal vector* $\mathbf{x}(t)$ (state space; (N) elements)
- the continuous scalar variable *time t*
- to a defined *output signal vector* $\mathbf{y}(t)$ (output space; (P) elements)

in a most general form. Thus, the (N) coupled, nonhomogeneous *ordinary differential equations* of first order and the (P) *algebraic equations* constitute a pair of the signal vectors $\dot{\mathbf{x}}(t)$ and $\mathbf{y}(t)$:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{f}\{\mathbf{x}(t), \mathbf{u}(t), t\} \\ \mathbf{y}(t) &= \mathbf{g}\{\mathbf{x}(t), \mathbf{u}(t), t\} \end{aligned} \tag{2}$$

with $\mathbf{f}\{\ldots;t\}$ and $\mathbf{g}\{\ldots;t\}$ as nonlinear, time continuous, time variant vector functions. This is the description of a *multivariable nonlinear, time variant* (NLTV) *dynamic process*; *time variant*, because the parameters depend on time *t*, and *dynamic*, because at least one relation is a differential equation.

Some simplifications deliver the popular standard description of the *multivariable linear, time invariant (LTI) dynamic process* of (N)th order [4] (Fig. 1):

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{x}(0) &= \mathbf{x}_0 \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{aligned} \tag{3}$$

with the initial state $\mathbf{x}(0)$

or in short

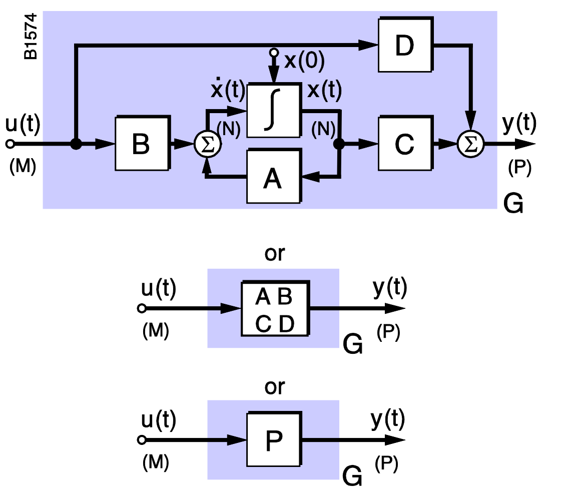

Fig. 1. Signal Relation Diagram (SRD) of a multivariable, linear, time invariant (LTI) dynamic system of (N)th order.

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \hdashline \mathbf{y}(t) \end{bmatrix} = \mathbf{P} \begin{bmatrix} \mathbf{x}(t) \\ \hdashline \mathbf{u}(t) \end{bmatrix}$$

with the partitioned, time invariant (constant) parameter matrix **P**

$$\mathbf{P} = \left[\begin{array}{c:c} \mathbf{A} & \mathbf{B} \\ \hdashline \mathbf{C} & \mathbf{D} \end{array}\right]$$

The *core* (kernel) of the description is the simple *homogeneous differential vector matrix equation* $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$. The (N) × (N) state matrix **A** of the state model exclusively represents the "dynamics" of the system. Since one is able to get to a model in different ways, the state matrix **A** is *not unique*. However, different state matrices **A** for one and the same process are *equivalent* and according to Linear Algebra equivalent matrices always have the same properties [14].

A *nondynamic* process is described exclusively by *algebraic* and/or *transcendent equations*. If we use the State Space Description again, only the *feed-through matrix* **D** remains: $\mathbf{y}(t) = \mathbf{D}\,\mathbf{u}(t)$. However, keep in mind that in the real world there is no nondynamic process. It is often a useful fiction or approximation within the *virtual world* of models.

Depending on the feasible combinations of cases, we finally arrive at five types of dynamic process descriptions of a state model:

– input–output model,
– input-state model,
– input-state-output model,
– state model,
– state-output model.

Each description type is used for dedicated analysis, statements and definitions concerning *properties* and *behaviour* of a process.

Up to now we have assumed constant parameters **p** within the system and have considered the *system transfer description* (called transfer function model) [2,4]: the $\mathbf{y}(t)$ depend on $\mathbf{u}(t)$. However, according to the cause and effect principle, we additionally have to include a second description of a system as soon as the parameters **p** vary over time and thus become independent variables. This type of description is called *system sensitivity description* (called sensitivity function model) [5]: the $\mathbf{y}(t)$ depend on $\mathbf{p}(t)$.

4. Properties and behaviour of the dynamic system

4.1. Properties

4.1.1. Properties of systems

Quantitatively, we describe *properties* of a system by *structures* of equations on the one hand and by vectors and/or matrices of *parameters* of these equations on the other hand. These are ascribed to the modelled process. We have already taken notice of a structural property of a dynamic system, the order (N), which denotes the number of derivative terms in the set of equations. Additionally, we have found a pair of terms as a parametric property, *time variant/time invariant*, which characterises the time dependence of the system parameters.

There are more such well-known properties, as for example:

– *Controllability*, a property, which is constituted by the *input-state model* (**B**; **A**)
– *Observability*, a property, which is constituted by the *state-output model* (**A**; **C**)
– *Linearity*, a property, which requires linearity of *all* relations

Note that *not* all properties must have a real, physical meaning.

4.1.2. Properties of signals

Concerning *signals*, we also ascribe to them individual time and space dependent *properties*, both with *structures* and with *parameters*. Note that properties of the input signal vector $\mathbf{u}(t)$ and the initial state vector $\mathbf{x}(0)$ have nothing to do with the properties of the dynamic system.

4.2. Behaviour

Each *system* will show a *behaviour*. But, what about *signals*? Generally speaking, every *signal* originates from some source, which we have called a *system*. We are not in a position to ascribe to signals an own, genuine *behaviour*, because we could not justify this by mathematical reasoning. We can only ascribe to them individual properties. This fact is essential for a consistent definition of terms concerning signals. Consequently, we will apply the term behaviour to systems only.

4.2.1. Verbal description of the behaviour of the dynamic system

A dynamic system *behaves* on different "inputs". On the one hand, it responses to arbitrary input excitations $\mathbf{u}(t)$ (stimulation, impact, action, effect, drive, load, question, disturbance) by its *transfer behaviour* and to arbitrary property deviations by the parameters $\mathbf{p}(t)$ by its *sensitivity behaviour*. On the other hand, it is depending on the chosen initial state values $\mathbf{x}(0)$.

Note: There is a conceptional difference between the *initial state* $\mathbf{x}(0)$, which is generally defined in the model and a certain *initial state value* x_0 at a certain time instance t_0. Given an *analytical* and/or an *empirical description* (model) of a dynamic process, its *behaviour* is known theoretically for any time t in the future and/or in the past. It can be predicted for the future, measured in the past, and analysed later on.

4.2.2. Mathematical description of the behaviour of the dynamic system

Mathematics gives us the analytical *general solution* and the infinite possible *particular solutions* of the set of equations (model) in the time domain as a function of time *t*. Accordingly, one particular *solution of equations* stands for one particular *behaviour of a dynamic system*.

The background: In the state model *time-derivative signals* $\dot{\mathbf{x}}(t)$of the *state signals* represent *generalised velocities* (rates of change) and intrinsically stand for *dynamics*. But normally, we are less interested in velocities.

The missing link up to now has been the procedure of *mathematical integration* of these time-derivative signals, in order to obtain the state signal vector $\mathbf{x}(t)$, that represents the *generalised positions*. Thus, *integration* enables *solution* in Mathematics and in System Theory. We call it the general or particular *behaviour*. For clarity reasons one is tempted to consider this mathematical operator "integration" as a metaphor of *accumulation* (charge, discharge) in storages, stores, capacitors, repositories, absorbers, accumulators, compartments, reservoirs, tanks, basins, memories, etc. Therefore, we consider each element $x_n(t)$ of the state signal vector $\mathbf{x}(t)$ a general *measure of content* of the integrator (n).

As to real circumstances, we define this statement as a *measure of information*, a *measure of mass*, a *measure of energy*, a *measure of momentum*, etc.

Consequently, the general and particular behaviour of the whole dynamic system (input-state-output model ($\mathbf{A}$; $\mathbf{B}$; $\mathbf{C}$; $\mathbf{D}$)) is represented by the set of solutions $\mathbf{y}(t)$, which had been generated from the set of differential and algebraic equations. The *output signal vector* $\mathbf{y}(t)$ of a linear, time invariant (LTI) dynamic system is the *general solution* and the *general behaviour*

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t) \tag{4}$$

which contains all parameter matrices. The parameter matrix $\mathbf{A}$, representing system dynamics, appears within a special time function, the *matrix exponential function* $\mathbf{\Phi}(t) = e^{\mathbf{A}t}$, called *state transition function matrix* [4].

The matrix exponential function $\mathbf{\Phi}(t) = e^{\mathbf{A}\,t}$ reminds us that the general solution of a multivariate, linear, time invariant (LTI) dynamic system contains as a *core* a multivariate superposition of (N) *exponential functions*, individually weighted by the elements of the parameter matrices concerned. We are already accustomed to this fact by the simple solution of the ordinary differential equation (system) of first order: $\varphi(t) = e^{a\,t}$.

Additionally, for a particular behaviour of a dynamic system we need its *initial state signal vector* $\mathbf{x}(0) = \mathbf{x}(t_0)$ at the chosen *initial time instant* t_0, which *synchronises* the model (system) with the modelled process, for example at the beginning of a synchronous simulation procedure. All state signals of the integrators are set to the state at this particular initial time instant t_0. Thereby, the *past* is included in the integrators. Note that the process itself has no initial state.

Note: It is an extremely useful advantage of the State Space Description that *all* linear, dynamic processes are described by the *same* structure in the time domain and in the frequency domain as well. This is true for the general solution (behaviour) too. Difficulties do not arise from seemingly challenging mathematical procedures but rather from our ability to develop an appropriate model of the process and from the quantitative identification of the parameters concerned.

Alternatively, we empirically determine the *behaviour of a dynamic system* by measuring provided and/or given input quantities $\mathbf{u}(t)$ on the one hand and resulting output quantities $\mathbf{y}(t)$ of the process on the other hand. An appropriate analysis of the acquired data develops the description (model) of the process of interest. We generally call this procedure *system identification*, or in Metrology *system calibration*.

If a *realisation* $u(t)$ of a *random ensemble* $\{u(t)\}$ enters a system, it will still remain a *deterministic dynamic system*, although the corresponding output realisation $y(t)$ of the *random ensemble* $\{y(t)\}$ is random. The same is true for *random initial values* $x(t_0)$. Only, if at least one of the parameters $\mathbf{p}(t)$ changes randomly, the dynamic system will become a *probabilistic, dynamic system*.

4.3. Static and stationary behaviour of a dynamic system

On the one hand, a *stable dynamic system* may reach a particular state, called the *static state* (steady state, rest state, equilibrium state). This is the case as soon as all input signals $\mathbf{u}(t)$ have become constant and therefore all time-derivative signals (generalised velocities) $\dot{\mathbf{x}}(t)$ in the differential equations have approached zero. By then, the system is described by the degraded set of differential equations. Consequently, for $t \to \infty$ only a set of algebraic and/or transcendent equations is left, the *dynamic state* will become a *static state*. Thus, this *particular description* and resulting *behaviour* is a very special case of the *normal description* and *behaviour*. Although this system is in the *static state*, it is still a *dynamic system*.

On the other hand, a *stable dynamic system* may reach a particular state, the *stationary state*. This is the case as soon as all input signals $\mathbf{u}(t)$ and thus their probability density functions $\mathbf{p_u}^{\mathrm{d}}(t)$ have become stationary and therefore all mean values $\boldsymbol{\mu}_{\dot{\mathbf{x}}}(t)$ of the time-derivative signals (generalised velocities) $\dot{\mathbf{x}}(t)$ in the differential equations have approached zero. By then the system is described by the degraded set of *differential equations concerning the mean values* $\boldsymbol{\mu}_{\mathbf{x}}(t)$. For $t \to \infty$ only a set of algebraic and/or transcendent equations is left. Thus, this *particular description* and *behaviour* is a very special case of the *normal description* and *behaviour*. Again, a time invariant system in the stationary state is still a deterministic, dynamic system.

We ascribe a *behaviour* of a *dynamic system* to the modelled *dynamic process*, fully aware that the correspondence is perfect only with probability zero. Deviations define *systematic behaviour errors* and *behaviour uncertainties* which are part of *systematic model errors* and *model uncertainties*.

5. Three important time dependent terms

The description of properties and behaviour of a dynamic system of the former section is the prime starting point for further definitions and terms in this field. We choose three typical property terms, which are particularly delicate, because at times it is unclear, how they have to be applied. The main requirement for a proper definition should be the availability of some mathematical basis.

5.1. The term stability

We frequently say that a procedure is unstable if a signal drifts away. Does this mean that *instability* and *drift* are synonymous terms? Not at all! The causes or the mechanisms of instability and drift in systems are completely different. There are *unstable systems* and *drifting systems*. A system can be stable and may still drift at the same time. Actually, this is even the normal situation.

Output signals, emerging from unstable systems, cannot be called unstable either. Instead, they are qualified as varying, fluctuating, oscillating, limit cycling, drifting, shifting, and as transient, nonconstant, nonstationary, unsteady, unbounded, and so on. Again: Such types of signals may stem from stable systems alike, due to dedicated input signals, which are able to generate such types of output signals just as well. Quantitatively, this is true for corresponding mathematical descriptions of output signals of course. There exist no definitions or mathematical formalisms for an "unstable signal"!

Therefore we concentrate on *systems* and define the *stability of a system* qualitatively as its ability (behaviour) to keep its output signals *bounded*, provided that all input signals are bounded too [2,13]. Once a system is unstable, at least one output signal will escape unbounded, even if all input signals and all system parameters remain bounded.

The quantitative definition of stability is simple for linear dynamic systems, but cumbersome for nonlinear systems. System Theory describes a linear time-invariant dynamic system (LTI) by a set of ordinary, linear differential equations (ODE) with constant parameters. When is it stable? The stability of a linear system depends on both system properties, namely on structures and on parameters.

1. System (model) *structure*: There are systems, which cannot become unstable for structural reasons, namely linear nondynamic systems and linear dynamic systems of less than third order. All other systems may become unstable.
2. System (model) *parameters*: The stability of a linear dynamic system of higher than second order depends on the *magnitudes of the parameters*, which are elements of the so-called characteristic equations and of the state matrix **A** of the system respectively. In the *time domain*, a linear dynamic system is asymptotically stable, if all poles (eigenvalues) of the set of *characteristic equations* of the *differential equations* have negative real parts and are therefore located in the left half of the complex plane. In the *frequency domain*, a linear dynamic system is asymptotically stable, if all poles (eigenvalues) of the *denominator polynomial* of the *spectral transfer function* $g(s)$ have negative real parts and are therefore located in the left half of the complex plane.

Further stability criteria can be seen from different points of views. The simplest are the two mentioned above. Mathematically they are equal and the statements are identical. Considering the magnitudes of the parameters, stability criteria deliver stability conditions, stability margins, stability bounds and stability areas, useful for system synthesis and system engineering.

Instability is a binary property (stable/unstable) and is based on system properties and not on signal properties, even though it is noticed by apparent output trajectories. We also have to avoid the frequently used term *long-term stability*, which concerns *drift* and not stability issues.

Besides, in Metrology, we seldom encounter unstable systems; in particular there are no unstable sensors. However, we sometimes meet unstable processes with implemented stable sensors. Sometimes incorporated reconstruction processes, observer processes, filter processes and adaptation processes raise stability questions in Metrology.

Note: An *unstable* dynamic system does not behave *chaotically*, since its behaviour is completely *deterministic* and can be predicted, provided the model of the system is known. Chaotic systems follow different rules.

5.2. The term stationarity

The term *stationarity* refers to signals (time series data) and parameters, but not to systems. There are several classes of stationarity. We only mention *weak stationarity*, which already shows the main characteristics of interest. Signals are (weakly) stationary concerning time, if characteristic *values* (like mean values, standard deviations values, covariance values and so on) and characteristic *functions* (like probability density functions, correlation functions, spectral power density functions and so on) are independent concerning time and space, or frequency and wavelength.

The simplest example refers to a stationary signal or parameter $x(t)$, for which the following equation (description) holds [3,10]:

$$\mathcal{AVG}\{x(t)\} = \mu_x = \lim_{t_{obs}\to\infty} \frac{1}{\Delta t_{obs}} \int_{t=-t_{obs}}^{+t_{obs}} x(t)dt = \mathcal{AVG}\{x(t)+\Delta\tau\} \tag{5}$$

with

$\mathcal{AVG}\{\ldots\}$ averaging (expectation) operator
μ_x arithmetic mean value
t_{obs} observation time value
$\Delta\tau$ arbitrary time shift interval

This means that for a stationary signal or parameter the arithmetic mean value is independent of time t. However, this seemingly simple definition asks for an infinite range of the independent variable, here of time t, which is never available. So, we normally define *short-time stationarity* within finite observation (averaging) intervals (windows) Δt_{obs} (Fig. 2).

Citation: The following source from the National Institute of Standards and Technology (NIST) may serve as an example of how delicate the use of common language terms can be, if it is not based on scientifically agreed conventions: "A manufacturing process cannot be released to production until it has been proven to be stable. Also, we cannot begin to talk about process capability until we have demonstrated stability in our process. A process is said to be stable when all of the response parameters that we use to measure the process have both constant means and constant variances over time, and also have a constant distribution." [8]. Obviously, stationarity is meant here, not stability.

5.3. The term drift

A rough characterisation in technical and natural sciences concerning *drift* claims that an observed signal or parameter changes on average in *time* versus a certain direction. When measuring or collecting data from a process, we may occasionally say: "Our data drift away". Besides, there is the related term *shift* too. It refers to a signal, which changes on average in *space* into a certain direction [11].

However, there are some consistent features:

– Within a dynamic system in steady state condition ($u(t)$ = constant or stationary) drift appears as a temporal deviation of at least one system parameter from the *nominal value* in a particular direction.
– Drift is perceived as a slow evolvement of behaviour compared to all other temporal patterns in the dynamic system.
– Regular results due to input–output *transfer behaviour* and potential drift due to *property sensitivity behaviour* may superpose. Several drift features may interfere with each other too.

Hence, drift, perceived in signals (data), is the special result of *temporal parameter variations* $\mathbf{p}(t)$ of a system and explicitly excludes results of varying input quantities $\mathbf{u}(t)$ of this system. Drifting deviations of system properties or system behaviour from stated nominal properties or behaviour are erroneous properties or behaviour. They give rise to *drift errors* $\mathbf{e}_{\mathrm{drift}}(t)$, accompanied by *drift uncertainties* $\boldsymbol{u}_{\mathrm{drift}}(t)$ (Fig. 3).

Since we assume constant or at least stationary input signals concerning the (*system transfer description* $u(t) \to y(t)$), we are allowed to concentrate, for the consideration of drift, on the second type of description mentioned above [5] (*system sensitivity description* $p(t) \to y(t)$). The term *robustness* characterises the *insensitivity* of the system against parameter variations.

At least one parameter, namely the drift parameter $p_{\mathrm{drift}}(t)$, changes in one direction. The parameter deviation

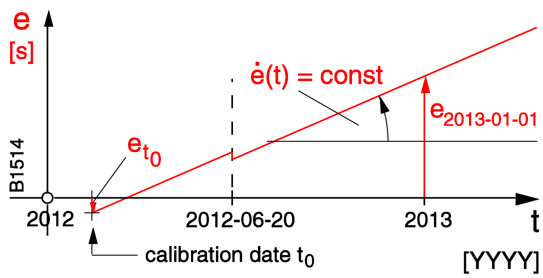

Fig. 3. Example of the systematic drift error e of an output signal of a drifting watch.

(error) $\Delta p_{drift}(t)$ makes the dynamic system a *time variant dynamic system*. We often assume that the drift proceeds linearly with time *t*: $\Delta p_{drift}(t) = v_{p_{drift}}\, t\ [\{p\}]$. The constant (average) drift velocity (drift rate) of the parameter *p* is $v_{p_{drift}} = \dot{p}_{drift}[\{p\}s^{-1}]$. It may change with time *t* too. Again, this is a description of system properties.

Output signals will show effects due to these drift properties. The simplest case is the nominal *constant output signal y*, which now drifts in time *t*. Thus, the signal is not constant any more. The nominal temporal trajectory y_{nom} is superposed by a temporal component $\Delta y_{drift}(t)$, which systematically tends in one direction.

$$y(t) = y_{nom} + \Delta y_{drift}(t)\ [\{y\}] \tag{6}$$

Assume a typical *stationary random output signal* y(t), whose nominal arithmetic mean value $\mu_{y_{nom}}$ drifts [10]. Because of the time dependence of the mean value, the random signal is *not stationary* anymore:

$$\mu_y(t) = \mu_{y_{nom}} + \Delta y_{drift}(t) \quad [\{y\}] \tag{7}$$

Note that the drift signal $\Delta y_{drift}(t)$ can be a random signal with a drifting mean value too, as soon as the parameter changes randomly with time.

So we always distinguish between the nominal trajectory (signal) $y_{nom}(t)$ and the deviating trajectory (signal) drift $\Delta y_{drift}(t)$.

These remarks are true for nondynamic (when existing) and dynamic systems. Drifting systems are regularly called *unstable*. This is *not* correct in the view of Signal and System Theory. In this respect the International Vocabulary of Metrology (VIM) [7] is misleading too with its definition 4.19: "stability of a measuring instrument: property of a measuring instrument, whereby its

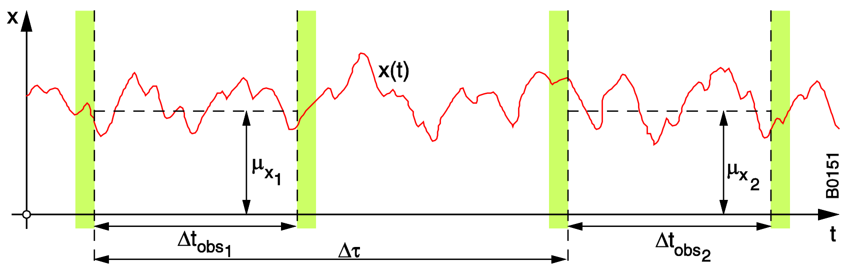

Fig. 2. Definition of short-time stationarity.

metrological properties remain constant in time" with a further quantification of this term: "In terms of the change of a property over a stated time interval." This would be the appropriate definition for a *time invariant system* and not of a *stable system* on the one hand and the suitable specification for a *drifting output signal* on the other hand according to system parameter drift.

6. Summary

In everyday language, *terminology* is not important. However, if we have to rely on it quantitatively in science and technology, in everyday practice, in literature, and in International Guides and Standards, it becomes important. Unsystematic terminology, ambiguity and misunderstanding can be avoided and thus communication can be simplified.

Several terms around the concept *dynamic system* and *time dependent signals* have been compiled and examined. Depending on the fields of natural and technological sciences, we meet diverse habits, meanings, terms, and definitions. Some are used arbitrarily and uncritically, some even dogmatically. Mainly mathematical principles have been applied here, to look for systematic and holistic terms for properties and behaviour. To some extent, terms had to be redefined. The aim was, to serve as many fields of science as possible. The following statements subsume the result:

- Both, dynamic processes and time dependent quantities are described by structures and parameters, which are properties of the describing systems and signals (models).
- The terms *dynamic/nondynamic* concern systems and processes respectively. They cannot be ascribed to *time and space dependent signals and parameters*, which always stem from dynamic systems.
- Constant parameters make a system *time invariant*; varying parameters make it *time variant*. The term *drift* characterises a special parameter varying system, whereby drifting signals arise.
- The term *static* describes a particular momentary *state* of a dynamic system. The antonym of a *dynamical system* is a *nondynamical system* and not a *static system*.
- The term *behaviour* concerns systems. The same is true for the terms *stable/unstable*. They are not applicable to *time and space dependent signals* and *parameters* (data). Signals of unstable systems move *boundlessly*. Unstable systems are rare in Metrology. They are not useful concerning practical applications. However, they can be stabilised by control tools.
- The terms *stationary/nonstationary* concern *signals* and *parameters* only. Signals are constant and stationary on the one side, and drifting, shifting, varying, fluctuating, oscillating, limit cycling, transient as well as nonconstant, nonstationary, unsteady, unbounded on the other side.
- Normally, different types of signals with different properties interfere with each other in a signal. Sometimes, a separation and analysis can be achieved by dedicated filter procedures.
- The concept «dynamic system», especially exemplified by the State Space Description for multiple input, multiple output (MIMO) systems, is generalisable in several directions by extension or reduction.
- The term *process* concerns all real world objects, in Metrology all sensor processes, data processes, error compensation processes, decision processes, auxiliary processes, and so on.
- Responses to this proposal of related terms are most welcome.

Selection of terms
system, model of a process
dynamic/nondynamic system
properties of a system
general/particular behaviour of a system
system in a dynamic/static state
deterministic/probabilistic system
time variant/invariant system
stable/unstable system
state transition function/sensitivity transition function of a system
initial state, initial state value of a system
signal, model of a quantity
time dependent/time independent (constant) signal (parameter)
stationary/nonstationary signal (parameter)
drifting/nondrifting signal (parameter)

Selection of symbols
Quantities and signals (parameters) are lower case
Vectors of quantities and signals (parameters) are lower case bold
Matrices of parameters are upper case bold
Index numbers in the text are marked by round brackets
Ensembles are marked by curly brackets
References are marked by square brackets
$\mathbf{u}(t)$ vector of input signals
$\mathbf{x}(t)$ vector of state (inner) signals
$\mathbf{x}(0)$ initial state vector/$\mathbf{x}_0$ vector of initial state values
$\mathbf{y}(t)$ vector of output signals
$\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ matrices of system parameters
$\boldsymbol{\mu}_{\mathbf{x}}(t)$ time dependent arithmetic mean vector of a vector ensemble $\{\mathbf{x}(t)\}$
$\boldsymbol{\sigma}_{\mathbf{x}}^2(t)$ time dependent variance vector of a vector ensemble $\{\mathbf{x}(t)\}$

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References

[1] A.V. Oppenheim, R.W. Schafer, Discrete-time Signal Processing, third ed., Prentice Hall, 2010.
[2] A.V. Oppenheim, A.S. Willsky, N.S. Hamid, Signals and Systems, second ed., Prentice Hall, 1997.
[3] A. Papoulis, S.U. Pillai, Probability, Random Variables, and Stochastic Processes, McGraw-Hill, 2002.
[4] W.L. Brogan, Modern Control Theory, third ed., Prentice-Hall, 1991.
[5] P.M. Frank, Introduction to System Sensitivity Theory, Academic Press, 1978.
[6] ISO, Evaluation of Measurement – Guide to the Expression of Uncertainty in Measurement (GUM), JCGM 100, 2008.
[7] ISO, International Vocabulary of Metrology – Basic and General Concepts and Associated Terms (VIM), third ed., JCGM 200, 2008.
[8] ISO, Statistics – Vocabulary and Symbols – Part 2, Applied Statistics, ISO 3534-2, 2010.
[9] K.H. Ruhm, Process and System – A Dual Definition, Revisited with Consequences in Metrology 13th IMEKO TC1-TC7 Joint Symposium, London, 2010, Journal of Physics: Conference Series 238 (2010) 012037, <http://iopscience.iop.org/1742-6596/238/1/012037>.
[10] K.H. Ruhm, Averaging One Variable, 2005 <http://www.mmm.ethz.ch/dok01/d0000547.pdf>.
[11] K.H. Ruhm, Drift, 2011 <http://www.mmm.ethz.ch/dok01/e0000603.pdf>.
[12] National Physical Laboratory (NPL), Teddigton, UK <http://www.npl.co.uk/server.php?show=ConWebDoc.2131>.
[13] NIST, Engineering Statistics Handbook <http://www.itl.nist.gov/div898/handbook/ppc/section4/ppc45.htm>.
[14] G. Strang, Differential Equations and Linear Algebra, Wellesley-Cambridge Press, 2014.